Solutions
Exam Probability and Measure (WBMA024-05)
Monday June 19 2023, 15.00-17.00

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, where the σ-algebra $\mathcal{A}$ is generated by the finite partition $\mathcal{P} = \{A_1, A_2, \cdots, A_r\}$ of Ω. Let $f : \Omega \to \mathbb{R}$ be an $(\mathcal{A}, \mathcal{B})$-measurable function. Show that there exist $\alpha_1, \cdots, \alpha_r \in \mathbb{R}$, such that f may be written as

$$f = \sum_{i=1}^{r} \alpha_i \mathbb{1}_{A_i}.$$

That is, show that f is constant on the separate elements of $\mathcal{P}$. (15pt)

SOLUTION: Suppose that there is $A_i \in \mathcal{P}$ on which f is not constant. Say that there are distinct $x, y \in \mathbb{R}$, such that $f^{-1}(x) \cap A_i \neq \emptyset$ and $f^{-1}(y) \cap A_i \neq \emptyset$ and therefore $f^{-1}(x) \cap A_i$ is a strict and non-empty subset of A_i. Which implies that $f^{-1}(x) \cap A_i$ is not in the powerset of $\mathcal{P}$ and therefore not in the σ-algebra generated by $\mathcal{P}$ (note that the powerset of a generating set always contains the σ-algebra generated by that set). Because $\{x\}$ and $\{y\}$ are in $\mathcal{B}$, this contradicts that f is an $(\mathcal{A}, \mathcal{B})$ measurable function.

2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, with $\mu(\Omega) = \infty$ and let $f : \Omega \to \mathbb{R}$ be an $(\mathcal{A}, \mathcal{B})$ measurable function. Assume $f \in \mathcal{L}^p(\Omega)$ for all $p \in [1, \infty]$ and that $\|f\|_\infty > 0$.

a) Provide the definitions of $\|f\|_\infty$ and $\|f\|_p$. (10pt)

Solution: $\|f\|_\infty = \inf\{c \geq 0 : \mu(\{\omega \in \Omega : |f(\omega)| > c\}) = 0\}$.
Or equivalently $\|f\|_\infty = \inf\{c \geq 0 : |f(\omega)| \leq c$ almost everywhere$\}$.

While $\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{1/p}$ for $f \in \mathcal{L}^p$

b) Show that $\|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p$. (10pt)

Hint: Take the integral of some useful function over $\Gamma = \{\omega \in \Omega : \|f(\omega)\| > M\}$, where $M \in (0, |f|_\infty)$. Relate this integral to $\|f\|_\infty$ and $\|f\|_p$ for given finite p.

Solution: We know $\|f\|_\infty > 0$, so we can choose $M \in (0, \|f\|_\infty)$ and set Γ as in the hint. By the definition of $\|f\|_\infty$ and by $M < \|f\|_\infty$ we have $\mu(\Gamma) > 0$. Now

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{1/p} \geq \left(\int_\Gamma |f|^p d\mu\right)^{1/p} \geq \left(\int_\Gamma M^p d\mu\right)^{1/p} = M\left(\mu(\Gamma)\right)^{1/p}.$$

Taking $\liminf$ on both sides gives that for all $M < \|f\|_\infty$ we have $\liminf \|f\|_p \geq M$ (where we only use $\mu(\Gamma) > 0$), which finishes the proof.

c) Formulate Hölder's inequality. (5pt)

Solution: If $p \in [1, \infty]$ and $1/p + 1/q = 1$ and $g_1 \in \mathcal{L}^p$ and $g_2 \in \mathcal{L}^q$, then $\|g_1 g_2\| \leq \|g_1\|_p \|g_2\|_q$.

d*) Show that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. (10pt)

Hint: Show that $\|f\|_p \leq (\|f\|_1)^{1/p} (\|f\|_\infty)^{1-1/p}$ and deduce that $\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty$. Combine this with part b).

Solution: Since from b) we know

$$\|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p,$$

it is enough to prove

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty.$$

Recall $\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{1/p}$. By Hölder's inequality, we know (using $g_1 = |f|$ and $g_2 = |f|^{p-1}$) that

$$(\|f\|_p)^p = \int_\Omega |f|^p d\mu = \int_\Omega g_1 g_2 d\mu \leq \|g_1\|_1 \|g_2\|_\infty = \|f\|_1 \||f|^{p-1}\|_\infty = \|f\|_1 (\|f\|_\infty)^{p-1},$$

where the last equality follows from the definition of $\|f\|_\infty$. So,

$$\|f\|_p \leq (\|f\|_1)^{1/p} (\|f\|_\infty)^{1-1/p}$$

Taking $\limsup_{p\to\infty}$ on both sides leads to the desired

$$\limsup_{p\to\infty} \|f\|_p \leq (\|f\|_1)^0 (\|f\|_\infty)^1 = \|f\|_\infty.$$

3. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $A_n \in \mathcal{A}$ for $n \in \mathbb{N}$, such that

$$\mathbb{P}\left(\bigcap_{i\in\mathcal{I}} A_i\right) = \prod_{i\in\mathcal{I}} \mathbb{P}(A_i) \qquad \text{for all finite } \mathcal{I} \subset \mathbb{N}.$$

That is, the A_i's are mutually independent. Let

$$\begin{aligned} A &= \{\omega \in \Omega : \omega \in A_i \text{ for infinitely many values of } i\} \\ &= \{\omega \in \Omega : \text{for all } n \in \mathbb{N} \text{ there is an } i \geq n \text{ such that } \omega \in A_i\}. \end{aligned}$$

a) Show that $A \in \mathcal{A}$. (10pt)

Solution: The definition of A translates to

$$A = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

Note that $\bigcup_{i=n}^{\infty} A_i$ is in $\mathcal{A}$ for all n, because a σ-algebra is closed under countable unions, and then $A \in \mathcal{A}$ because a σ-algebra is closed under countable intersections.

b) Show that if $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty$, then $\mathbb{P}(A) = 1$. (10pt)

Hint: You may use $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$.

Solution: Note that

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^C) = 1 - \mathbb{P}\left(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (A_i)^C\right) \geq 1 - \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{i=n}^{\infty} (A_i)^C\right).$$

Then we note that for any $n, n' \in \mathbb{N}$ with $n' > n$ we have

$$\mathbb{P}\left(\bigcap_{i=n}^{\infty} (A_i)^C\right) \leq \mathbb{P}\left(\bigcap_{i=n}^{n'} (A_i)^C\right) = \prod_{i=n}^{n'} \mathbb{P}\left((A_i)^C\right)$$

by the independence of the A_i's and thus the independence of the $(A_i)^C$. So,

$$\mathbb{P}\left(\bigcap_{i=n}^{\infty} (A_i)^C\right) \leq \prod_{i=n}^{n'} (1 - \mathbb{P}(A_i)) \leq \prod_{i=n}^{n'} e^{-\mathbb{P}(A_i)} = e^{-\sum_{i=n}^{n'} \mathbb{P}(A_i)},$$

where we have used the Hint in the second inequality. Note that because $\mathbb{P}(A_i) \leq 1 < \infty$ we have that $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty$ implies $\sum_{i=n}^{\infty} \mathbb{P}(A_i) = \infty$ for all $n \in \mathbb{N}$. Because $\mathbb{P}\left(\bigcap_{i=n}^{\infty} (A_i)^C\right)$ does not depend on n' and $\mathbb{P}\left(\bigcap_{i=n}^{\infty} (A_i)^C\right) \leq e^{-\sum_{i=n}^{n'} \mathbb{P}(A_i)}$, for all $n' > n$, we obtain by $n' \to \infty$ that $\mathbb{P}\left(\bigcap_{i=n}^{\infty} (A_i)^C\right) \leq e^{-\sum_{i=n}^{\infty} \mathbb{P}(A_i)} = 0$ and therefore,

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^C) \geq 1 - \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{i=n}^{\infty} (A_i)^C\right) = 1 - \sum_{n=1}^{\infty} 0 = 1.$$

Let C be a circle with circumference 1 (i.e. with radius $1/(2\pi)$). One by one randomly chosen closed arcs (on C) denoted by I_i ($i \in \mathbb{N}$) of respective lengths ℓ_i are colored red. Assume further that $1 > \ell_1 \geq \ell_2 \geq \cdots$. Denote the midpoint of I_i by x_i. The x_i are chosen independently and uniformly on C. Let S be the part of the circle that is colored red. That is,

$$S = \cup_{i=1}^{\infty} I_i.$$

Note that the uniformity of the x_i's implies that for every $c \in C$ and $i \in \mathbb{N}$ we have $\mathbb{P}(c \in I_i) = \ell_i$. Define

$$k_n = 2^{n+1} \times n! \quad \text{and} \quad K_n = \sum_{i=1}^{n} k_i \qquad \text{both for } n \in \mathbb{N}.$$

For $K_{n-1} < i \leq K_n$ let $\ell_i = 1/(2k_n)$. So, there are k_n arcs of length $1/(2k_n)$.

c) Show that $\mathbb{P}(c \in S) = 1$ for all $c \in C$, and that C will eventually be red almost everywhere (with respect to Lebesque measure on C). (10pt)

Solution: Fix $c \in C$. Define $A_i = \{c \in I_i\}$, so $\mathbb{P}(A_i) = \ell_i$. Note that (with $K_0 = 0$),

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \sum_{n=1}^{\infty} \sum_{i=K_{n-1}+1}^{K_n} \mathbb{P}(A_i) = \sum_{n=1}^{\infty} \sum_{i=K_{n-1}+1}^{K_{n-1}+k_n} 1/(2k_n) = \sum_{n=1}^{\infty} 1/2 = \infty.$$

and apply part b to obtain that c is even in infinitely many I_i.

Now assume that we would have picked $c \in C$ randomly according to Lebesque measure (that is uniformly), then still $\mathbb{P}(c \in S) = 1$, so S covers C almost everywhere.

d*) Show that $\mathbb{P}(C = S) < 1$. (10pt)

Hint: You may use the following approach:
Show that with positive probability, there is an infinite sequence of non-empty arcs $a_1 \supset a_2 \supset \cdots$ of respective lengths $1/(2k_1), 1/(2k_2), \cdots$, such that $a_n \cap (\cup_{i=1}^{K_n} I_i) = \emptyset$. You can do this by first showing that a_1 exists with strictly postive probability and then condition on that a_n exists. Then split a_n in $k_{n+1}/(2k_n)$ disjoint arcs of length $1/k_{n+1}$, and show that with "desirable" probability at least one of those arcs does not contain any of the x_i for $i \leq K_{n+1}$.
You may use without further proof that there exists $z \in (0,1)$, such that out of any subset of j disjoint arcs of length $1/k_{n+1}$ the probability that at least one of those arcs does not contain any of the x_i with $K_n < i \leq K_{n+1}$ is larger than $1 - z^{-j}$ for all $n \in \mathbb{N}$.

Solution: Follow the hint: Partition C in k_1 arcs of length $1/k_1$. The probability that one of those arcs does not contain any of the points $x_1, \cdots, x_{k_1}$ is strictly positive (exact probability is not important). Denote this probability by p_1. Choose one such arc and call it b_1. Because the distance of a point in I_i to x_i is at most $\ell_i/2$, if b_1 exists, it contains a subarc a_1 of length $1/(2k_1)$ such that $a_1 \cap (\cup_{i=1}^{K_1} I_i) = \emptyset$.

Now suppose that we have a_n of length $1/(2k_n)$ such that $a_n \cap (\cup_{i=1}^{K_n} I_i) = \emptyset$. Then Partition a_n in $k_{n+1}/(2k_n) = n+1$ disjoint arcs of length $1/k_{n+1}$. The probability that at least one of those subarcs contains none of the x_i for $K_n < i \leq K_n + k_{n+1} = K_{n+1}$ is by the last part of the hint at least $1 - z^{n+1} =: p_{n+1}$ for some $z < 1$. Arguing as for a_1, the probability that a_{n+1} exists given that $a_1, \cdots a_n$ exist is p_{n+1}.

The probability that a_n exist on every level $n \in \mathbb{N}$ is bounded from below by $\prod_{n=1}^{\infty} p_n$, which for every $N \in \mathbb{N}$ is equal to $\prod_{n=1}^{N} p_n \times \prod_{n=N+1}^{\infty} p_n$. Then,

$$\prod_{N+1}^{\infty} p_n = \prod_{n=N+1}^{\infty} (1 - z^n) \geq 1 - \sum_{n=N+1}^{\infty} z^n = 1 - \frac{z^{N+1}}{1-z},$$

which is strictly positive for large enough N and the question is answered.